

Introduction to Block 1

There are a good many ideas that are introduced in the first block and are developed and extended in subsequent blocks. Chief among these is the notion that algebra provides a language in which to express conjectured generalities. The power of the symbolic language is that it enables those symbols to be manipulated.

1

First encounters with expressing generality

Chapter 1 introduces one of the central themes of the book:

- Every learner who starts school has already displayed the power to generalise and abstract from particular cases, and this is the root of algebra.

The suggestion made is that expressing generality is entirely natural, pleasurable, and part of human sense-making. Algebra provides a manipulable symbol system and language for expressing and manipulating that generality. The core pedagogic issue is therefore about enabling learners to employ their natural powers in using algebra to make sense both of the world and of other people's use of algebra.

The first three sections of the chapter look at generalities experienced in patterns in numbers, patterns in diagrams, and patterns outside school. Section 1.4 returns to number patterns and suggests things to do when you are stuck on a mathematical problem. Section 1.5 looks at the ideas of the chapter from a pedagogic point of view.

In the quickies in this chapter, you will be asked to write down some numbers. It is important that you do actually write them down, for there is a significant difference between imagining the numbers, and actually writing them down.

1.1 EXPRESSING GENERALITY IN NUMBERS

Quickie 1.1

Write down a number that is 1 more than a multiple of 10. Write down another. Write down another. What are their remainders when you divide them by 10?

Comment

Variations of this task will recur during this chapter. As with all tasks in this book, what matters most is not your specific answer but your response, that is, what you notice about yourself in attempting the task.

What did you notice about how you went about the task? Did you simply write down suitable numbers? Did you find that you were beginning to be more adventurous when you wrote the third number? Were you surprised when you found the remainder?

Were you able to use the experience of the first two to help with the third? Do you have a sense of something that is or might be 'always' true? How confident are you about it?

Almost certainly you found yourself choosing which multiple of ten to add to 1 in order to produce a number, though you may not have thought precisely in those

terms. Nevertheless, you have a sense of a generality. Whilst generalising is natural, learners need time to notice when they have a sense of a generality, and time to express generality, to strengthen and extend their natural powers to generalise.

Number Patterns

One of the most important sources of generalisations is the domain of number, in detecting and expressing number patterns.

Task 1.1.1 Anything plus Anything

A young child observed that $3 + 5 = 5 + 3$, that $2 + 4 = 4 + 2$, and that 'anything plus anything is anything plus anything'. Express in your own words what you think the child was probably trying to express.

Comment

As it stands, the statement could be interpreted as, for example, taking 'anything' as independent, so that for example $3 + 5 = 17 + 22$, yet it seems entirely implausible that the child was thinking this way. The more likely interpretation is that if you select any two numbers, then you get the same answer, no matter in which order you add them.

The child's statement is an expression of generality, transcending the particular instances and dealing with an infinite number of possible cases. The actual expression of generality may, as in this case, need refining to avoid misconstruing it. Often the first attempt to express a generality turns out to be too wide-ranging, which is why any such expression needs to be treated as a conjecture, as something that needs checking in particular instances, and justifying in general, perhaps at some later date. Algebra as language provides ways of being more precise in the expression of generality and, later, for reasoning about properties of numbers.

Task 1.1.2a Mental Multiplication

If you had to multiply 32 by 18, in your head, how might you go about it?

Comment

If you *had* to do it in your head, you might think either of 32 as $30 + 2$ and multiply both by 18 before adding, or else think of 18 as $20 - 2$, multiplying both by 32, and then subtract appropriately. You might know another way.

It is clear that learners are not expected to memorise all possible calculations with numbers conforming to patterns like $(30 + 2) \times \text{something}$. Rather, they are expected to discern a 'general method'. The next task considers possible variations on this theme.

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Task 1.1.2b What can be Changed?

What can be changed in the form $(30 + 2) \times \textit{something}$, and still be of help in doing mental multiplications?

Comment

Perhaps the most salient variation is in the tens: $(30 + 2) \times \textit{something}$ suggests $(40 + 2) \times \textit{something}$, $(50 + 2) \times \textit{something}$, and so on. Awareness of the possible change is awareness of a general method, a generality. Furthermore, there is already an implicit generality in changing the other number, the 18, to *something*: it does not matter what the second number is, the multiplication is likely to be easier when spilt up than when attempted directly. Furthermore, it can be useful if '+ 2' is replaced by '+ 1', or '+ 3', or, depending on mental facility, 4 or even 5. Instead of adding 6, or more, any number could be decomposed as a subtraction of 1, 2, 3 or 4. There are three different features of the original task that can be changed: the tens number, the 'extra bit', and the number to be multiplied by.

Task 1.1.2c What is the Generality?

What implicit generality is being exemplified by the previous methods of doing a multiplication?
Try your articulation out on others to see if they can make sense of it.

Comment

It is quite hard to state a generality succinctly but clearly, even after several attempts. One attempt at a generality might be 'you find a number close to one of the numbers that is easier to multiply by'. Another attempt might be: 'mental multiplication can be done more easily by linking at least one number to a nearby number that is easier to compute with'. Notice that the previous sentence already includes an implicit generality: 'linking at least one number to a nearby number' is not specific, but general. To learn to use this 'mental method' is to come to appreciate and be familiar with a generality.

It is not easy to express the generality, partly because there are several 'things' that could be varied. If care is not taken, you end up like the child in Task 1.1.1, using general labels such as *something* or *anything* to refer to different items.

Not all generalities need to be expressed explicitly, but it is worth noting how often there is an underlying generality that learners are expected to appreciate and to be aware of, even if not explicitly. Learners need plenty of experience of expressing generalities in order to make sure they are appreciating generalities implicit in techniques and concepts, rather than just trying to reproduce those techniques.

The next task demonstrates a general form of task for provoking learners to use their natural powers to generalise.

Task 1.1.3a In Sequence

Fill in the fourth line in this sequence of arithmetic statements:

$$(3 + 2) \times (3 - 2) = 3^2 - 4$$

$$(4 + 2) \times (4 - 2) = 4^2 - 4$$

$$(5 + 2) \times (5 - 2) = 5^2 - 4$$

(6

Task 1.1.3a In Sequence

Is it correct? What do you think comes next?

Comment

Most people know what is coming next: ‘+’ then a ‘2’ then a bracket, and so on. And the next line and the next line ... Those three dots are called an *ellipsis*, meaning a ‘short-form’, and were introduced by Isaac Newton to mean ‘and so the pattern continues’. But what is that pattern?

When working on tasks such as this, there is often an assumption that the learners ‘see what the teacher sees’. Indeed, since many people do not recognise ‘seeing’ as a description of what they do in their heads, even that assumption needs to be rephrased: ‘Learners do not always attend to what the teacher is attending to’.

If some people are ‘seeing’ or attending to different features than those seen or attended to by the teacher, they may find what the teacher says quite mystifying. It is important for learners to have time to think about, formulate and try to articulate generalities, to themselves and to each other, before expressing to a group or a whole class. At first, it is necessary to call upon them to generalise explicitly in a variety of different contexts. As the practice of becoming aware of generality develops, and as expressing generality becomes a habit, it may not be necessary to call explicitly for generality every time.

Therefore the next task is offered as an introduction to expressing generality.

Task 1.1.3b In Sequence (continued)

Complete the rows in Task 1.1.3 that start (37 ... and (987654321

Comment

If you are not sure, write some more rows that follow on from the ones given, not just writing numbers, but paying attention to what you are changing and what you are preserving, that is, to what stays the same. A useful label for this process is *Watch What You Do*. By asking yourself ‘what is the same about each row?’ and ‘what is different and how is it changing?’ you can focus attention usefully in order to detect generalities in the form of patterns of relationships.

The brackets are as much a feature of the sequence as are the numbers, and deserve explicit attention. Work with brackets is developed explicitly in sections 3.3 and 7.3.

Rina Zazkis (2001), a researcher in mathematics education, has found that using very large numbers, which are intentionally daunting when it comes to performing arithmetic calculations on them, is a good way to prompt learners to become aware of generality.

Task 1.1.3c In Sequence (continued)

Someone far away is thinking of a number, and because you don’t know what it is, it is necessary to denote it by something such as a little cloud ☁, to show that someone is thinking of it. Their number is the first number in one of the rows. What does the rest of the row look like? The row starts (☁).

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Comment

To display the line with clouds in this text would block you from doing the real work, which is detecting the structure of the rows and expressing it, probably by becoming aware of how you knew what to write down in the previous task. In words, ‘cloud plus two’ times ‘cloud minus two’ is cloud squared take away 4.

As you move from row to row, your eye naturally discerns things that are the same every time, and things that change. For example, each line starts with a left bracket. The symbol sequences ‘) × (‘and’) =’ are common to every line. The symbol also appears to the right of the = sign in each case. By stressing what is changing, and consequently ignoring what stays the same, you invoke your natural power to detect something that is changing and can take different values or qualities, thus experiencing generality.

Indeed, what may be most striking at first is the familiar number sequence 3, 4, 5 However, the significant pattern is found, not by following the sequence of counting numbers one by one, but by looking at what is being said on each line: ‘The product of “anything” plus 2 and ‘that same thing’ minus 2 is the difference between their square and 4’.

Notice how hard it is to articulate the generality in words, because of the problem of referring to something general more than once. Using clouds makes it clearer, as long as it is agreed that what is in the cloud is fixed (the person does not change the number they are thinking of just yet).

$$(\text{☁} + 2) \times (\text{☁} - 2) = \text{☁}^2 - 4$$

One advantage of the use of clouds with numbers is that it makes the generality much less open to ambiguity. The statement can be ‘looked at’, but it can also be read: ‘two more than the number you are thinking of, times two less than it, is the square of that number minus 4.’

Task 1.1.3d In Sequence (continued further)

Now what happens if the number another person is thinking of in their thought cloud is 13/3? Does ‘it’ still work? What is the ‘it’?

Comment

What is the range of permissible change for the ‘anything’, that is, for the entry in the cloud? You may already be convinced that it can be any counting number; introducing a fraction is intended to suggest that it might work for *any* fraction. In which case, why not *any* number? What if you use the square root of 10? One of the purposes and powers of algebra is that it makes it possible to locate and describe succinctly the range of permissible change within which a relationship holds: in this case, two expressions *always* yield the same value.

In every case so far, two expressions such as

$$(3 + 2) \times (3 - 2) \text{ and } 3^2 - 4$$

have been equal in value – different ways of expressing the same number. Mathematicians like this sort of regularity, so when they encounter less familiar numbers in the same computation, they look for the pattern to continue. The next task shows how this can lead to some discoveries.

Task 1.1.3e In Sequence (continued backwards)

What happens if you work your way backwards four or more rows?

$$(4 + 2) \times (4 - 2) = 4^2 - 4$$

$$(3 + 2) \times (3 - 2) = 3^2 - 4$$

(2

(1

Comment

The row beginning '(2' is a reminder that multiplying by 0 must give 0. The row beginning '(1' can be interpreted as saying that $3 \times (-1) = 1^2 - 4 = -3$, which, generalised, leads to 'multiplying a positive by a negative gives a negative'. The row beginning '(0' can be interpreted as saying that $2 \times (-2) = -4$. The row beginning '(-1' can be interpreted as saying that $1 \times -3 = (-1)^2 - 4$, so since by the pattern of the previous rows you would expect $1 \times (-3)$ to be -3 , it must be the case that $(-1)^2 = 1$ in order to preserve the pattern based on the equality of different expressions.

This is the fundamental reason why it is generally accepted that $(-1) \times (-1) = 1$: it allows the 'rules of arithmetic' to extend beyond positive whole numbers to negative numbers.

Pause for Reflection

The most important notion in this section is that when a pattern is detected in which some things are changing and others staying the same, there is an opportunity to express a generality, in this case, $(n + 2) \times (n - 2) = n^2 - 4$.

This stated relationship is, for the moment, only a conjecture, because there is only a sense of pattern, an intuition of how numbers work, on which to depend. (How do you know whether it works when, say, $n = 987654321$?)

In the next few sections, the notion of expressing generality will be exemplified in contexts other than number patterns.

Task 1.1.R Reflection

What struck you about the work in this first section? Make a note of ideas that you would like to come back to you when preparing for or conducting a lesson.

Comment

Possibilities include:

- pleasure in expressing generality;
- the discovery of properties of numbers such as multiplying negative numbers together arising from expressing generality arising from patterns;
- thinking about what can change whilst some relationship remains invariant.

There are likely to be other things as well.

Comment on tasks

Remember that if you find a task too simple or too familiar, you have an opportunity to devise your own, similar task that stretches and challenges you. One way to do this is to ask what you might change and still have a task that calls upon the same kind of thinking. For example, in Task 1.1.3, some of the invariants in the expressions are the two 2s on the left, and the 4 on the right. Suppose someone suggested changing both 2s to some other number: what then would replace the 4?

1.2 EXPRESSING GENERALITY IN DIAGRAMS AND PICTURES**Quickie 1.2**

Write down a number that is 2 more than a multiple of 5. Write down another. Write down another. What are their remainders when you divide them by 5?

Comments

Did your thinking at the start of the last section come back to you?

This section continues the theme of expressing generality, in the context of reading diagrams and pictures. It calls upon your power to imagine change.

Generalising from Diagrams

The following task requires you to imagine a relevant diagram in each case.

Task 1.2.1 What is General About ...?

What is general about each of the following statements?

The sum of the angles of any triangle lying in the plane is 180° .

If you know the three lengths of sides of a triangle, you can construct the triangle uniquely.

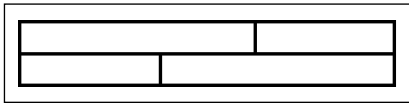
Comment

In the first statement, the triangle can be *any* triangle whatsoever, however extreme.

In the second statement, the three lengths must belong to a triangle, but otherwise are perfectly general.

Often the generality in a statement is hidden in language such as *a* or *any*. The word *any* can be taken to refer to a specific object, or to an arbitrary choice of object, hence to all such objects.

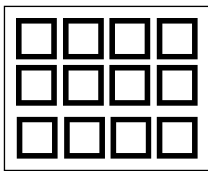
Opportunities to generalise arise in using simple diagrams. For example, if you have two line-segments or blocks and you put them in line, with the front end of one at the back end of the other, you get a new segment whose length is the sum of the two segments. Furthermore, you can put either of the two segments first, and then the other:



Children learn that the two compound rods will be the same length. Articulating that awareness of the way the world works expresses a generality that is one of the rules of arithmetic: you can add two numbers in

either order; the result is always the same.

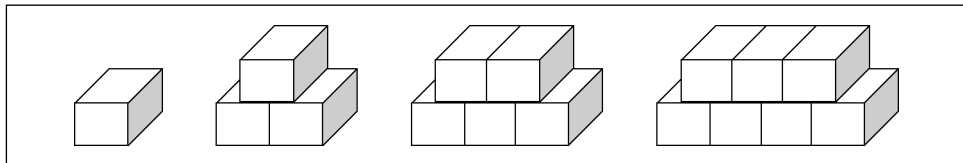
A similar fact applies to multiplication. To count the number of squares in the following array, you can count how many in each row, and the number of rows, or how many in each column and the number of columns.



Generalising from Picture Sequences

Opportunities to generalise arise when a sequence of objects is being counted.

Task 1.2.2a Brick Walls

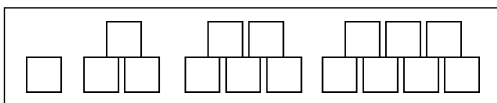


Decide how you are going to continue the picture sequence. The best way to do this is to make or draw some more yourself, first in order to clarify your general rule, and then in order to become aware of *how* you are counting. Once you can say in words how the sequence extends, you have your first expression of generality.

Comment

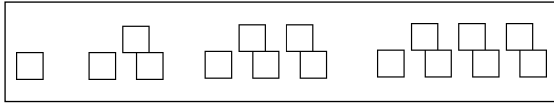
Once you have specified a way of extending the sequence of pictures indefinitely using a rule, there is a unique answer to counting the number of bricks needed to make any particular wall (such as the thirty-seventh one, which is a typical two-digit number). There are usually many different ways of seeing *how* to do the counting, which therefore provide opportunities for choice and creativity.

The point of counting in individual cases is to become aware of *how* you are counting. Sometimes it is easier to draw a picture to *show how* you are counting, rather than to say it in words. For example, here are two of the many different ways of seeing the brick wall sequence: ‘I see one brick, with none, then with one pair, then with two pairs, . . . of bricks added on.’



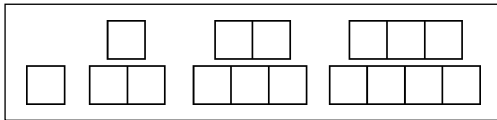
or more deliberately, to emphasise the adding of pairs of blocks, like this:

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Notice that an alternative perception might be: 'I see two rows, the top row having one brick less than the bottom row.

The bottom row has one, two, three, . . . bricks in it, depending on its sequence position.' An alternative is 'the bottom row has as many bricks as the picture number; the top row has one fewer'.



Each way of seeing gives rise to a way of counting, which can be expressed more succinctly (as your confidence grows), until it looks very much like a formula. Thus:

Picture number 37 will have two rows of bricks.
There will be 37 bricks in the bottom row and 36 bricks in the top row.

This can, when appropriate, be shortened to:

Picture 37 needs 37 (bottom row) + 36 (top row) bricks.

Algebraic thinking has already begun. By the time facility has been achieved in being able to find the number of bricks needed in particular cases, sufficient attention is freed to see and express generality.

Task 1.2.2b How Many Bricks? (generalised)

Someone has a picture-number in their head. Tell them how to calculate how many bricks that picture has.

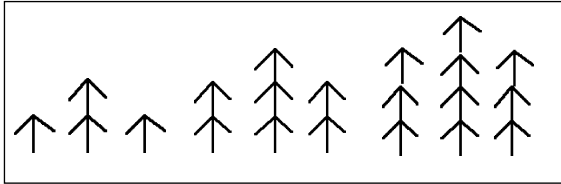
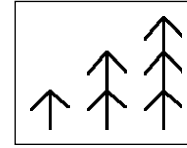
Comment

This is the heart of algebra seen as expressing generality. Where does the generality lie? The point of this task is to lead up to seeing how to count the number of bricks in general, by seeing a structure or pattern in how each and every picture is constructed.

Let the person's picture number be denoted by a cloud. Then there will be ☁ in the bottom row, and $(☁ - 1)$ in the top row, or $+(☁ - 1)$ altogether. Alternatively, there will be 1 plus a multiple of 2, namely $1 + 2(☁ - 1)$, or again, 1 less than a multiple of 2, namely $2☁ - 1$.

When you express a generality, such as in the last task, it is important to be clear on the status of your expression. Is it a conjecture, or do you have some way to justify it as being always correct? Reference to the structure, such as 'two rows of picture-number bricks then remove one from the top row' constitutes a justification. However, it is always wise to test a conjecture, even an almost certain conjecture, on one or two typical examples, as there can be a slip between 'seeing' or 'having a strong sense of' the structure, and actually expressing that accurately in symbols.

Picture sequences worth counting can come from many sources, such as children's own drawings or traditional designs from different cultures. They can also arise spontaneously at unexpected moments.

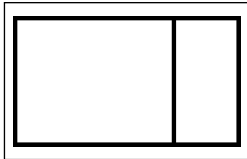


For example, while producing a Christmas picture, children in one class were drawing trees. They soon realised they could make bigger and bigger versions, and that they could count the twigs, or the twig-ends, not just for particular trees, but for any such tree no matter how large. They could even develop forest clumps growing in some regular manner, such as shown here.

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Reading Areas

A single diagram can usually be interpreted as illustrating a whole range of possibilities, a generality.



In the diagram, the total area of the large rectangle can also be expressed as the sum of two areas. The sum of the areas of the two small rectangles is the area of the large rectangle. Because no statement has been made about the actual areas, this is a very general statement. An algebraic version of the generality runs along the lines of:

if the height is h and the widths are a and b respectively,
then the total area is both $h(a + b)$ and $ha + hb$.

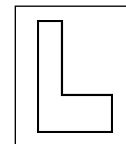
Although they look different, these two expressions must therefore be equal, at least as long as a and b are possible lengths. No matter how you move the pieces, area is conserved.

This is, of course, the reason why mental multiplication strategies such as those used in Task 1.1.2 always work. The following tasks provide more experience of using multiple expressions.

Task 1.2.3a Finding Areas

Find at least three different ways to decompose this shape into rectangles to find its area.

You have to decide on appropriate lengths for yourself, but the aim is to express a method that works in general, for all 'L-shapes'.



Comment

Did you find yourself wanting to use some specific numbers, or letters, to describe your method or did you use shapes?

For complicated shapes, even in the plane, it is not always easy to discern how to break a shape into rectangles in a maximally efficient manner. Learners often struggle with seeing how to break up a shape into rectangles in order to find its area.

Task 1.2.3b Finding Complicated Areas

Make up a figure from rectangles and draw its perimeter, hiding the component rectangles. Clearly, someone else could partition this shape into rectangles. Now make up a more complex shape that is harder to partition with a minimum number of rectangles.

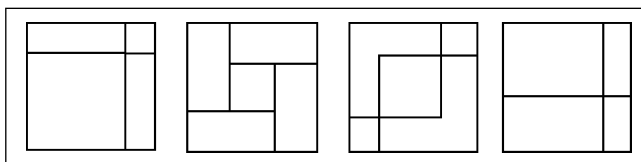
Comment

The principle behind this task is that in order to gain facility in discerning component rectangles, it is useful to make up your own.

Restricting attention to squares partitioned into squares and rectangles yields useful experience in discerning features, recognising relationships, and expressing generality concerning different ways to calculate total area, as the next task shows.

Task 1.2.4 Reading Areas

Interpret each of the diagrams as general statements about areas in as many different ways as possible.



Comment

The first thing you have to do is to 'see' the diagram as made up of components. Selecting different features will lead to different readings.

Paying attention to the range of values that a variable can take is an important part of introducing symbols, because it is easy to forget necessary restrictions. If you add letters, a and b , for the sides of the two inner squares, a and b represent lengths and must therefore be non-negative. The first diagram can be interpreted as $(a + b)^2 = a^2 + 2ab + b^2$. If a denotes the size of the large square and b one of the inner squares (again with a being a non-negative number, and b a number between 0 and a), then $a^2 = (a - b)^2 + b^2 + 2b(a - b)$.

Pause for Reflection

Reading a diagram or picture is actually very similar to reading a symbolic expression:

- you have to discern details that contribute to the overall picture;
- you look for relationships between elements, and sometimes also between diagrams or expressions;
- you express those relationships;

- you ask yourself what can change in this diagram and still the detected relationships hold;
- you identify relationships that can be taken to hold for a whole class of diagrams, expressions, or other objects, and in so doing you construct properties.

Sometimes these happen so quickly that they slip by unnoticed; sometimes learners get stuck because they have not shifted appropriately, or they are stressing different features.

Relationships hold between elements, so it is necessary to stress or foreground those elements, and as a result, to ignore or background other elements or features. Caleb Gattegno (1970), an influential educator and philosopher, suggested that the process of stressing some features and ignoring others is, in itself, the process of generalising.

Task 1.2.R Reflection

What comes to mind as you think back over the work in this second section?

What differences do you find between generalising from a diagram or a sequence of diagrams, and generalising from patterns in numbers? Which do you feel most comfortable with?

1.3 EXPRESSING GENERALITY OUTSIDE SCHOOL

Quickie 1.3

Write down a number that is 3 more than a multiple of 7. Write down another. Write down another. What are their remainders when you divide them by 7?

Write down a 10-digit number that is 3 more than a multiple of 7.

Write down a number that is 3 more than a multiple of 7 and which it is unlikely anyone else reading this book will think of writing down (yes, you *can* do it).

Comments

While carrying out this task, you might have noticed that when asked to construct something, then another, then another in quick succession, you find, that by the time the third one is requested you are starting to be a bit more creative, a bit more extreme (Watson and Mason, 2004).

When, in class, learners hear others describe their third number, they sometimes find themselves thinking ‘why didn’t I think of that?’, or even, ‘I would never have thought of that!’ But in truth, it takes only a few exposures to these sorts of tasks for most people to begin to relax and explore more extreme cases. For example:

Write down a number with 10 digits that leaves a remainder of 1 on dividing by 10, by 5, by 7.

(Note that 7 might require more thought.)

Write down a number you think no one else in the room will think of writing down.

Write down a number that is very large.

The purpose here is to help learners to become aware that in any situation, when asked for an example, they have a variety to choose from. It is not a matter of selecting the first that comes to mind, but of pausing to consider the entire set from which to make a choice. A choice can be made to be simple, really simple, quite complicated, or really complicated. In making this choice, awareness of the infinite extent, and yet of having a method of constructing any one of them, sets the stage and lays the groundwork for expressing generality.

This section offers a variety of tasks intended to alert you to the pervasive need for expressions of generality in the world outside of school. Customers want to know what something will cost them. Entrepreneurs need to devise pricing policies such as sale discounts and bulk purchase discounts, as well as deciding what mark-up to put on different items. Consequently, customers and entrepreneurs use algebra thinking, even if they do it by using a calculator or a spreadsheet.

Using Symbols

Since using words is both tricky and tedious, especially if you have to keep writing them all down, mathematicians use labels for objects they want to speak about. Most people learn, long before their first lesson in algebra, that ‘algebra is about using letters’. Unfortunately, many people never discover what those letters are for, or why they are used. Since it is important for learners to grasp the notion of being able to talk about and manipulate ‘whatever number someone else is thinking of, even if it is not yet known’, the cloud of section 1.1 provides a useful transitional tool between the informality of words and the formality of letters. After a period of time using clouds, the notion of using a single letter is likely to arise quite spontaneously as a reduction in effort.

Task 1.3.1 From Words to Symbols

For each of the following relationships, express the generality in words and symbols:

The height of a plant at the end of some specified number of days if it grows by 2 cm per day.

The number of days in a specified number of weeks.

The number of minutes in a specified number of hours; specified number of days; specified number of weeks.

The number of fence posts and rails needed to make a 3-rail fence along a stretch of road.

Comment

Make sure that you state clearly that whatever symbol you use, it stands for the number of something or other! How can you check if your conjecture fits the situation? Look out for some similar opportunities when shopping or reading the newspaper.

Whenever a generalisation is first expressed, its status is one of conjecture: it is an attempt to express something. However, it is very likely that there are flaws, either in the expression, or in the insight that is being expressed. Mathematical thinking can only take place within a supportive atmosphere, in which everything said (including everything asserted in this book!) is taken as a conjecture to be tested out in experience.

Task 1.3.2a Value Added Tax (VAT) and Discount

In a discount warehouse, you see an item advertised at 20% off the listed price, but you know that there is a VAT of 17.5% to be added on. Which would you prefer, to have the discount taken off first before the VAT is added on, or to have the VAT added on and then the discount taken off the total? Which do you think the customs and excise would prefer?

Task 1.3.2a Value Added Tax (VAT) and Discount (continued)*Comment*

Did you try a particular example, perhaps a simple example such as £1 or £100. You might even decide to make the VAT a more tractable quantity such as 10%, thereby revealing an implicit conjecture that the answer does not depend on the actual discount and tax.

A few calculations will probably produce a conjecture that for the customer, it does not matter: either order gives the same result. But why?

Start with £500. To discount by 20% means to subtract off 20% of the original: $(500 - (20\% \text{ of } 500))$, which is the same as $(100\% \text{ of } 500) - (20\% \text{ of } 500)$, which is the same as 80% of £500. So a discount of 20% applied to 500 is the same as taking 80% of £500, and this generalises by replacing the 500 by any other sum, and the 20% and 80% by any discount and its difference from 100%.

This is an example of implicitly or intuitively ‘seeing the general through the particular’. Trying a particular case using objects with which you are familiar and confident is entirely natural. This is what is meant by *specialising*. The purpose of specialising is to move to a simple or simpler particular case in order to ‘see what is going on’, in order to generalise. Sometimes the simplest possible case can be informative, or perhaps some other extreme and special case, but sometimes you need a less specialised case in order to see what is going on.

An excellent way to investigate the effects of discount and VAT is to use a spreadsheet, on which each step of the calculations can be displayed. A change in one of the values produces an instant change in the answers, enabling learners to experience a number of examples in quick succession, thereby supporting them in expressing generality. This generality remains a conjecture until the equality is proved generally to be true and not just special to a few cases, but the spreadsheet provides an ideal way to do many examples very quickly.

The formulae are displayed here.

	A	B	C	D	E	F
1	VAT	0.175	Price	500	Price	=D1
2	Discount	0.2	Reduction	=D1*B2	VAT	=B1*F1
3			discounted price	=D1-D2	price inc VAT	=F1+F2
4			VAT	=B1*D3	reduction	=F3*B2
5			Actual price	=D4+D3	Actual price	=F3-F4

The process of telling cells what calculation they are to do using the contents of other cells is a step towards expressing generality.

Task 1.3.2b VAT and Discount (continued)

Express a generality of which ‘so a discount of 20% is the same as taking 80%’ is a special case.

What role does the £500 play in the calculations? Express a generality.

Do something similar for ‘adding VAT’.

Express a full generality together with an explanation of why the generality is valid.

Comment

Calculating a discount of $D\%$ means calculating $(100 - D)\%$ of the price. Denoting VAT by V , adding in VAT means calculating $(100 + V)\%$ of the price. So, you are looking for values of D and V for which the following two expressions are the same:

$(100 + V)\%$ of $(100 - D)\%$ of the price, and $(100 - D)\%$ of $(100 + V)\%$ of £100.

But compounding percentages is multiplication, and it does not matter in what order the multiplication is done. So what the customer actually pays will be the same, whatever the order. This is clearly indicated in the spreadsheet, where no matter what values are inserted for the price, the VAT or the Discount, the Actual price is always the same, no matter in which order it is calculated.

For the customer, the order does not matter. For Customs and Excise, and for the entrepreneur, however, it matters a lot. Customs and Excise get more if the VAT is calculated on the full price rather than on the discount. However, the entrepreneur would lose out. Of course, this is unreasonable, so Customs and Excise declare that VAT is the last thing to be calculated, after all other calculations are done.

Task 1.3.2c VAT and Discount (continued)

The statements about Customs and Excise and the entrepreneur in the previous paragraph are examples of general statements. Do some examples to check these statements. In other words, try some special cases, first to appreciate what the generalities say, then to check that they are correct in particular cases, and then to see why the general statements must always be true.

Suppose a warehouse offers a discount of $D\%$ and furthermore, no VAT to be paid by the customer. What is the effective discount being offered?

Comments

Try some examples with a view to generalising!

Other Contexts**Task 1.3.3 Sale!**

The following price reductions have been sighted in shops. Put them in order of increasing reduction:

- Two for the price of one.
- Buy one, get one free.
- Buy one and get 25% off the second.
- Buy two, get one free.
- Buy two and get the cheaper one free.
- Buy two and get 50% off the second one.
- Three for the price of two.



Make up your own variants and then put them all in order of reduction!

Task 1.3.3 Sale!*Comment*

One approach is to try them all out on an item of say, £100. Another is to work out what fraction of the original price is being charged for each one.

Behind the arithmetic there are generalities lurking: no prices have been mentioned, so these apply to any prices; 'buy one' and 'buy two' suggest 'buy n ' where n is some, probably small, positive whole number; 'get one free' and 'get one at reduced price' suggest that you might get two or three at the reduced price (or free), and the reduction could be anything from 100% off (i.e. free) to 0% off (i.e. no reduction).

Task 1.3.4 Customs and Excise

How does a shop work out how much VAT they owe Customs and Excise if all their shop prices are quoted as inclusive of VAT?

Comment

The question is posed in general, so a sensible approach is to try particular cases, and then to look through the particular in order to express the general.

Pause for Reflection

Sorting out discounts and percentage increases requires a cool head, and is informed by checking the calculations on specific numbers. To express the generalities, it helps to use symbols like $D\%$ for discount percentage and $V\%$ for the VAT percentage as an aid to remembering what they represent. Often it is actually clearer to use symbols than to use numbers, because it is then possible to read the expressions. The symbolic expressions are both a statement of what calculation to do, *and* the answer to the calculation!

Pursuing an investigation that requires the learner to process a lot of examples is a far more satisfying way of getting practice than doing a set of exercises.

Task 1.3.R Reflection

What struck you about the work in this section?

Have you begun to notice occurrences of or opportunities for generalising outside of your teaching institution?

1.4 MORE EXPRESSING GENERALITY**Quickie 1.4**

Write down an expression for all numbers that leave a remainder of 1 when divided by 10; all those that leave a remainder of 2 when divided by 5; all those that leave a remainder of 3 when divided by 7.

This section considers arithmetic topics that sometimes cause some confusion when treated purely as arithmetic, but which are actually easier to think about when approached algebraically, by generalising.

Remainders

Since remainders cause some confusion, especially when used with negative numbers, this subsection begins by generalising the quickies, and then addresses the question of what it means mathematically to find the remainder on dividing -3 by 5 .

Task 1.4.1a Remainders and Multiples

Write down a number that leaves a remainder of r when divided by m . Now write them 'all' down, as a general expression,

Generalise the observation that 2 and 7 have the same remainder when divided by 5.

Comment

Trying particular cases is a good way to get a sense of what is going on and appreciating the structure, leading to an expression of generality. Quickie 1.4 was meant to provide that background. Trying particular cases is also important for testing conjectures, but can never validate a conjecture, unless you try all possible cases. One of the main points of algebra is that you can 'try' an infinite number of cases symbolically.

Task 1.4.1 can be generated by asking yourself what features of the quickie can be varied. The second part is another way of looking at the same situation as part one: two numbers have the same remainder on dividing by m if their difference is a multiple of m .

The point of tasks such as this is to prompt generalisation, so that learners become adept at answering 'questions of this type', that is, that they become familiar with the whole class of questions of which this is a representative.

Task 1.4.1b Remainders and Multiples (continued)

Write down in order, left to right, all the numbers between -13 and 12 that are 2 more than a multiple of 5.

Look for and express to yourself a relationship between successive pairs of your numbers.

Now starting from the right, write down underneath each number its remainder when divided by 5.

What is the same and what is changing? Generalise.

Comment

The positive numbers all leave a remainder of 2 when divided by 5. What does it mean to ask for the remainder when a negative number is divided by 5? The most sensible thing is to expect that it will continue to be the smallest positive number that, when subtracted from the number, makes a multiple of 5. So -3 is 2 more than -5 that is a multiple of 5, so the remainder on dividing -3 by 5 is 2. Remainders always lie between 0 and the divisor.

For numbers of the form $5n + 2$, the remainder on dividing by 5 is always 2. Consequently it makes sense, following this pattern, to define the remainder on dividing n (a positive or negative integer) by a number m to be the number r where $0 \leq r < m$ and $n = qm + r$. The q is called the quotient. In other words, r is the smallest non-negative number you can add to some multiple of m to get n . The two descriptions, one in terms of adding on and the other in terms of remainders, then describe the same thing.

Task 1.4.1c More Remainders

What is a reasonable meaning to give to the remainder on dividing -3 by -5 ? What about the remainder on dividing 2 by -5 ?

Comment

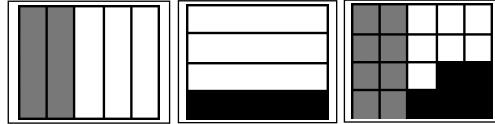
Numbers of the form $-5n + 2$ will have remainder 2 on dividing by -5 , so the remainder on dividing by 5 and by -5 is the same, and more generally, the remainder on dividing by n and by $-n$ is the same.

Fractions**Task 1.4.2a Fractionated**

What fraction of the first whole rectangle has been shaded in?

What fraction of the second whole rectangle has been shaded?

What fraction of the third whole rectangle has been shaded (light, dark, both)?

*Comment*

Many learners struggle with adding fractions. There are many reasons for this, including what fractions mean, why they are needed, cultural antipathy to fractions, and attempts to teach methods without a firm appreciation of images to fall back on.

This task suggests an image that doubles as a method either for adding fractions or for exploring fractions in order to work out a method of adding them. It is important when contemplating using any mediating tool, whether software, diagram or metaphor, to be clear about what learners need to do in order to make effective use of the tool.

Task 1.4.2b Generalising Fractionated

What features of the first rectangle diagram correspond to the fraction shaded as $2/5$ (one reading) and as $8/20$ (second reading)? In other words, what must someone discern and relate in order to be able to read the diagram as depicting a fraction? What features must all three rectangles share?

How is the third related to the first two?

Comment

It is vital to be explicitly aware of how the fraction is read, in order to be able to decide what rectangle to draw for yourself when given a fraction. Note that reading someone else's diagram and depicting a fraction for yourself are reverse operations. Depicting can be assisted if you are aware of how you go about reading.

Note that great care is needed when using diagrams to represent fractions. The light-shaded rectangles are depicted as a fraction of one rectangle, the dark-shaded are depicted as a fraction of a second, and it is only when they are depicted as fractions of the same whole that you can sensibly add them.

Arithmetic Rules

In Task 1.1.2, it was pointed out that a mental arithmetic strategy for multiplying numbers close to tens is to decompose the number and multiply separately. Another version of this same rule was then illustrated using area diagrams prior to Task 1.2.4. There are several mental strategies that can be hard to verbalise but easy to recognise when the situation arises.

Task 1.4.3 Mental Strategies

The following general statements are taken from Augustus de Morgan writing in 1883.

For each statement, construct an example in which it might be useful, then express it in symbols as a generality.

'We do not alter the sum of two numbers by taking away a part of the first, if we annex that part to the second.'

'We do not alter the difference of two numbers by increasing or diminishing one of them, provided we increase or diminish the other as much.'

'If we wish to multiply one number by another, we may break up one of them into parts and multiply each of the parts by the multiplier, and add the results.'

'The same thing may be done with the multiplier instead of the multiplicand (the number being multiplied).'

'If any two or more numbers be multiplied together, it is indifferent what order they are multiplied, the result is the same.'

'In dividing one number by another, we may break up the dividend, and divide each of the parts by the divisor, and then add the results.' ... 'The same thing cannot be done with the divisor.' (De Morgan, 1883 pp. 23–4).

Comment

Notice that in the second statement, 'increasing or diminishing' means adding or subtracting, not multiplying or dividing.

Did you construct an example to show that 'the same thing cannot be done with the divisor'? Finding counter-examples to variations is an important component of appreciating what a statement is really offering.

One way to work on these observations with learners is to use diagrams or to use a standard situation that learners can imagine. For example, imagine you have a bag of marbles, but you do not know exactly how many there are. You can remove some marbles with one hand, and some others with the other hand. If you remove one less with the first hand, and one more with the second, you still remove the same number over all.

THOANs

The acronym **THOAN** is a short form for **THink Of A Number**, where numbers are here taken to mean any number with which learners can comfortably do arithmetic, usually whole numbers. A calculator may be useful for trying specific numbers or for

extending the range of numbers. In various forms, THOANs have been used since medieval times. Adolescents are often intrigued by a mathematician's ability to predict answers after seemingly complex sequences of calculations. Of course, it is all done with algebra.

Task 1.4.4a THOANs

Think of a number between 1 and 10. Add 1; double the result; add 3; subtract 4; add 5; halve the result; add 6; subtract 7; add 8; subtract 9; subtract the number you first thought of.

Comment

Your answer is 1, no matter what number you started with.

One way to THOAN is to imagine you are moving about on a number-line. Another way is to imagine the numeral in your mind.

Of course, you can vary the ending by asking people for their answer and then telling them their starting number, or subtracting 1 and telling them they now have their starting number.

How can the answer be independent of the starting number? Although each participant starts with a particular number, the leader is ignorant of what they have chosen, and so starts with a symbol that stands for an unknown starting number.

To see why, write down the number you first thought of (a particular number). Now write down each subsequent calculation to form a list, one under the other, as an arithmetic operation but *without* actually doing the arithmetic. (It is important that you actually do this!). Now look through your statements and check for ambiguity. Is it perfectly clear which calculation is to be done first, second, and so on in each line? You will need to use brackets correctly, according to the rules discussed in Tasks 1.1.2 and 1.2.4.

Beside your chosen number in a second column, write a cloud. Now on each subsequent line rewrite the calculation using cloud in place of your chosen number. Be careful not to replace the numbers used in the instructions with clouds. Now do what arithmetic you can with the numbers in your final expression. It will eventually come down to 1, showing that no matter what number you started with, represented by cloud, it all disappears in the end to leave you with 1.

Task 1.4.4b THOAN (generalised)

What aspects of the particular calculation sequence could be varied?

How simple a THOAN could you design?

Comment

After enjoying a sequence of THOANs her brother was playing on a car journey, a 7-year-old girl asked if she could 'do one'. She started in confidently and then realised she did not know what to do! So she decided to make a simple one. After a few 'add three, now subtract three', she alighted on 'Think of a number; that's the number you thought of!' with a variant: 'Think of a number; subtract the number you just thought of; your answer is zero!'. She laughed and laughed.

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Other THOANs include

THOAN; add 2; multiply by the number you first thought of; add 1; take the square root; subtract the number you first thought of; your answer is again 1.

THOAN; square it; add 4; subtract four times the number you first thought of; take the (positive) square root; add 2; you're left with the number you first thought of.

You can easily make up your own; the more complicated they get, the more dependent you will become on being able to manipulate the expressions in order to simplify them. One of the features of a good THOAN is that it builds up a complicated expression one way, and then undoes the expression in a non-obvious way.

Pause for Reflection

Some important arithmetical ideas have passed by along the way to using the expression of generality in order to make sense of those ideas.

Task 1.4R: Reflection

What aspects of generalising are unclear or problematic for you?

Are there any topics you teach that do not involve generalisation in some way?

Comment

It is not always easy, or even possible, to articulate what it is that you do not understand. It is easy to say 'I don't understand' in a very general way, but much more useful to try to articulate what it is specifically. If a conjecturing atmosphere has been developed, so that learners are willing to try to express themselves even when they have a tentative suggestion, then it is much easier to be of assistance than if they wait until they are confident before saying something out loud.

Every mathematical topic, by virtue of being mathematical, involves generalisation. For example, a technique is a 'general method' for resolving a class of similar problems; a concept is a generality that has many different exemplars. Thus, every lesson affords opportunities for learners to generalise for themselves.

1.5 PEDAGOGICAL ISSUES

This fifth section highlights some of the observations made in the context of specific tasks offered during the previous four sections. Starting with some thoughts about the quickies, working through issues concerning the nature of algebra, it ends with a reflection on the structure of the tasks used in the chapter.

Quickies

The quickies started out in the form 'one more than a multiple of ...', and moved into the language of remainders. The form of the task one might use with specific learners depends on their competence. For example, with some year 7 pupils at the beginning of the year, Jackie Fairchild¹ formulated the task as in the first part of Task 1.5.1.

Task 1.5.1 Coined Tables

I have a 2p coin, and a large (unlimited?) supply of 5p coins. What values can I make?

How do those values relate to the five times table?

How do they relate to the quickies in this chapter?

Comment

Most year 7 pupils quickly recognise that you can make 2 plus a multiple of 5, and suggest representations such as $2 + [\text{something}] \times 5$. The 'something' could be a box, a cloud (to represent a number that someone is thinking but as-yet-unknown to us), a cloud (to represent anything that someone might choose to think of), a word or short-form (number or numb) or a letter (such as n for number).

One way to familiarise learners with exercising their powers is to use variations on the same task over a period of time.

Powers

You saw in some of the earlier tasks (for example, Task 1.3.2) that specialising means trying out particular cases of some generality, in order to develop a sense of what is going on. Whenever a generality is encountered, it is valuable to try to 'see the particular in the general', that is to seek specific, confidence-inspiring, familiar examples. Specialising is something people do all the time in conversation, especially when they try to offer examples or instances that contradict or support, challenge or extend what is being asserted. Every child who gets to school has already displayed the power to specialise for themselves, spontaneously.

Generalising is the flip side of specialising. It means locating some generality that encompasses a collection of particular cases, which is what the tasks have been about. Whenever a particular object or collection of objects is encountered, it is worthwhile trying to 'see the general through the particular', by asking what can be changed while leaving the idea or technique or problem much the same.

What is Algebra?

This chapter has proposed that algebra is most usefully seen as a language in which to express generalities, usually to do with numbers. Learners will only understand algebra as a language of expression if they perceive and express generalities for themselves. At first this takes time, but in this way learners become effective users of algebra.

Experience with expressing generality in different contexts leads to multiple expressions for the same thing. For example:

- the area diagrams (see Task 1.2.4) are open to several different readings leading to different expressions;
- counting features of picture sequences such as the number of blocks, edges, faces, etc. almost always leads to many different ways of seeing how to count, and each of these yields a different expression of the same generality: the count;
- the sales tax and discount tasks (see Tasks 1.3.3 and 1.3.4) show that there are different ways of calculating tax and price, and care is needed to make sure that the correct calculation is being used.

For each of these examples, the general expression reveals much more than the particular cases.

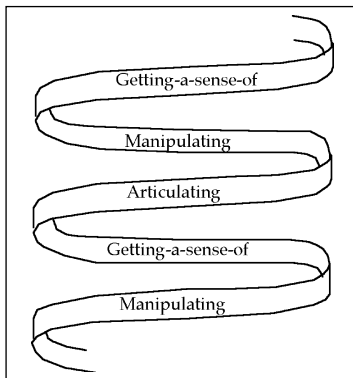
Pedagogy of the Chapter

The chapter, like most of the book, is structured around mathematical tasks. It is what you notice while doing the tasks that is likely to inform your future use of algebra and your future teaching of it. The sections have been constructed with a number of pedagogical constructs or principles in mind. This provides one important reason that you should actually do the tasks yourself, rather than just thinking about them or imagining yourself doing them – that ‘doing’ is the only means by which you can experience the pedagogic ideas directly for yourself.

For example, it has been suggested that manipulating familiar objects that inspire confidence is the beginning of getting a sense of structure, and that the structure eventually emerges in the form of a generalisation or expression.

You will learn most from your own first articulations of the emerging generalisation. Recognise that all first articulations are likely to be flawed, sometimes seriously. They are best treated as conjectures, for their status is temporary and conjectural. By saying conjectures out loud, and even making a brief note of them, you separate yourself somewhat from them, so you can look at them more dispassionately. By externalising them, using your notebook, they can be pinned down long enough to test them, and perhaps modify or even reject them in favour of a new version.

These ideas are usefully summarised in terms of manipulating–getting-a-sense-of–articulating (MGA) as a cycle or spiral of activity.



At any time you can specialise to some example that is familiar and confidence-inspiring, manipulating it so as to get-a-sense-of structure leading to re-articulating the general for yourself. Sometimes the first specialising is insufficient, so further specialising is required.

The MGA spiral is thus intimately related to the use of the power to specialise and to generalise that drives mathematics.

You may have noticed yourself, or learners, using ‘it’ quite often when trying to articulate a generality. For example, ‘you take it and you add to it and then you double it’.

In this case, the third ‘it’ refers to something different from the first two. It is well worth while becoming attuned to learners using ‘it’ when explaining something, and inviting them to clarify to what the ‘it’ refers, for often ‘it’ hides confusion or obscures a switch in what is being attended to.

Task Strategies

Some strategies have been suggested for prompting learners to become aware of possible generalities. These include:

- paying attention to *how* you draw or calculate or count in particular cases (*how* particular cases work) in order to become aware of the *how* as a generality;

- using large unwieldy numbers that no one wants to calculate with in order to direct attention towards structure and away from particular calculations;
- paying attention to and probing the use of indefinite pronouns ('it', 'that', 'this') when learners are explaining what they are doing or thinking (often 'it' and 'this' hide a slide from one object to another, creating confusion);
- explicit use of variation in detail to promote a shift of attention from particulars to what can be varied or changed and still the same idea applies (see for example, Task 1.1.2a);
- ... another and another ... often leads people to expand their horizons and become more creative, even extreme, in the construction of mathematical objects, thus enriching their awareness and their experience;
- encouraging the learner to construct their own extreme or complicated examples to reinforce their creativity, and in becoming aware of a range of possibilities from which to choose, to develop their sense of generality. Being trusted to make choices (and to check conjectures) supports learners in seeing themselves as active, competent doers of mathematics.

Most of the task-types used in this book have been borrowed from the culture of classrooms and from numerous researchers and teachers, such as Brown and Walter (1983), Prestage and Perks (2001), Watson and Mason (2002; 2004) and Bills *et al.* (2004).

It is vital to challenge learners, but not to over-challenge them with tasks they cannot do, or under-challenge them with tasks that are too easy. The same applies to your work on this book: you should feel able to work on tasks that challenge you, extend those that are easy for you and decide that some tasks are inappropriate for you at this time.

You may feel that the pacing of this chapter has not been quite right for you – perhaps too fast or too slow. Either way, further examples can be found on the associated website (<http://cme.open.ac.uk/algebra>). The pedagogic significance of the chapter lies in becoming attuned to noticing opportunities, within standard topics, to pause briefly and get learners to express generality.

Final Reflections

Task 1.5.R Reflection

What struck you about the work in this chapter?

What do the following terms mean to you, currently? Try telling someone else, or writing down something about them.

Specialising

Generalising

Expressing generality

Conjecturing

Algebra

What aspects of the tasks gave you some pleasure (even just a tiny bit)? Not which tasks, but what happened inside you? How might that happen for your learners?

Note

- 1 Private communication. This was brought to our attention by Jackie Fairchild.